Global games on social networks*

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Abstract

We propose a global game model situated within a social network of information exchange to examine how network topology and information structure influence strategic participation in a collective action. Central to our contribution is the establishment of the conditions for equilibrium uniqueness. The subsequent comparative statics outline the effects of shifting model parameters on player behavior. Our analysis explores the interplay between a player's trust in their prior beliefs and information received from peers, and its impact on their actions. We extend this framework by introducing a network formation game and using simulations to illustrate the evolution of social ties under various edge-formation approaches. Our analysis demonstrates how the emergence of social bubbles and echo chambers can alter information flow, leading players to make decisions that decrease their overall utility.

Keywords: global games, social networks, comparative statics, network formation, simulation.

JEL Codes: C71, C73, D7, D8, Z1

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1 Introduction

In recent years, we have been noticing several large-scale protests and revolutions being coordinated through social media. Some notable examples would be the Arab Spring, the anti-government protests in Russia, or the Black Lives Matter movement (Alqudsi, 2012). At the same time, events such as the GameStop stock short squeeze originating from Reddit showed that such coordinated events can happen fully online and achieve their goals (Chohan, 2021). Even activities such as voting for a non-mainstream president in an election can be viewed from the perspective of the information that players receive in a binary, collective action (Allcott and Gentzkow, 2017). However, despite in-depth coverage of these actions, the underlying mechanisms were never described in the form of a decision model.

The methods used to model situations where a large number of players are taking a binary action - to participate in an event or not - did not take the information sharing into account. We change that by proposing a model of a global game in which players interact with each other over a network that they create. We start by analyzing a model of a global game on a static network and provide a general condition for the uniqueness of an equilibrium in such a game. We later show how it changes as we allow for an endogenous network and how actions taken by players are affected by network topology and its evolution.

Traditionally, in global games, each player obtains a private, noisy signal about a state of the world and forms expectations about potential payoffs and behavior of others based on that signal (Carlsson and van Damme, 1993). This type of game is used to model events ranging from protests (Angeletos et al., 2007; Angeletos and Werning, 2006), through investments (Morris and Shin, 2001), to bank runs (Atkeson, 2000; Morris and Shin, 2000). In this paper, we take the framework of a global game and extend it by adding a mechanism for players to exchange their private signals on a network.

Our approach differentiates the uncertainty about the state of the world that each player faces by having private signals supplied by their neighbors. Our contribution to the current literature on global games on networks (Dahleh et al., 2012; de Martí and Milán, 2019; Mahdavifar et al., 2018) is an evolution of the Morris and Shin equilibrium and their conditions for the uniqueness of the equilibrium. Our model allows for information exchange by players, generalizing the game and its applications. We confirm the robustness of our results by proving they are resilient to heterogeneity in players' beliefs as in (Izmalkov and Yildiz, 2010), general beliefs updating mechanism (Brandenburger et al., 1992), and a non-vanishing noise. The other approach to global games on networks was presented by Leister et al., 2021. They combine mechanics from a global game with payoffs dependent only on one's neighbors. Despite differences between the models, we debate similar cases of networks.

The protests and revolutions mentioned at the outset are some examples of actions coordinated via social media, where information sharing proved essential to their growth. The volume and speed of information reaching potential participants directly shape their decisions to join and their level of commitment (Boulianne, 2019; Sturm and Amer, 2013). Furthermore, the specific social ties through which this information flows significantly impact the choice to participate in collective action (Enikolopov et al., 2016). Despite their importance, these factors have been largely overlooked in existing research. Our model addresses this gap by incorporating social network topology and analyzing how the strength of connections and the weight of shared information versus prior beliefs influence player behavior.

Our model is generic and allows for further generalizations and adjustments. The proof for the uniqueness of the equilibrium can be transferred to different types of information exchange protocols and network topologies, as shown in the appendix of this paper. We also work on dynamic payoffs as opposed to de Martí and Milán, 2019 and continuous beliefs, contrary to Barbera and Jackson, 2020.

The global games literature characterizes equilibria through two primary analytical approaches. First, as the variance of players' private signals diminishes to zero, equilibrium actions can be determined via iterative elimination of dominated strategies (Morris and Shin, 2000). Second, equilibria are derived through the concept of Perfect Bayesian Equilibrium (PBE), where optimal strategies are defined by threshold beliefs that govern whether players undertake risky actions (Oyama and Takahashi, 2020). Our model is constructed to ensure that these distinct analytical frameworks produce equivalent equilibrium results.

After establishing the proof for a unique equilibrium, we provide comparative statics to demonstrate how variations in expected payoffs, information exchange parameters, and network topology shape both the equilibrium and player behavior. This analysis builds upon and extends the existing literature on games with strategic complementarities (Van Zandt and Vives, 2007; Vives, 1990). Unlike many established network games where social effects impact payoffs or costs directly (Argenziano, 2008; Galeotti et al., 2010), our model isolates network effects to the formation of player beliefs. Furthermore, we offer new insights into games with incomplete information (Bergemann and Morris, 2013) and illustrate the specific role network architecture plays in determining outcomes (de Martí and Zenou, 2015). Finally, we examine scenarios where the uniqueness condition is not satisfied, using specific network structures to illustrate these complexities.

Social networks are rarely random; instead, they reflect an individual's prior choices and existing beliefs (Currarini et al., 2009; Ferrara, 2012). Both social media and general social interactions are known to drive polarization in political discourse and societal attitudes (Hong and Kim, 2016). Because people actively shape their own networks, the information they encounter is often a direct result of their previous actions (Bail et al., 2018). This process frequently leads to "echo chambers"—groups where individuals with similar views reinforce one another, pushing their collective beliefs toward extremes (Garimella et al., 2018; Oliveros and Várdy, 2015).

To examine how these echo chambers and other network phenomena evolve, we extend our model to allow players to build their own networks. We analyze which connections form when players optimize for their payoffs within a global game. Our study compares several edge-formation strategies: first, a random connection model similar to (Acemoglu and Jackson, 2017); second, a mutual consent mechanism where both parties must agree to a link (Acemoglu et al., 2014); and third, a "group-based" approach common on social media platforms like Facebook. In all variants, we account for the cost of establishing new connections and measure how varying these costs influences final player decisions (Fabrikant et al., 2003).

We identify conditions under which echo chambers become the most probable outcome of our proposed game. Importantly, the effects described here extend beyond scenarios like protests or revolutions. The mechanisms of belief formation and endogenous network creation we establish provide a framework for analyzing broader phenomena, such as public choice and the evolution of social norms (Centola, 2010; Jackson and Yariv, 2007).

Based on our endogenous network model, we run simulations to show how social networks evolve and to illustrate the effects described in our theory. Our results show that echo chambers form most often among players with extreme private signals. As the network grows, the cost of participation increases for those taking action. Both the model and the simulations suggest that information exchange within echo chambers can discourage players from choosing a "safe" option they would otherwise prefer. Instead, these structures often confirm existing biases, pushing players toward costly actions they would have avoided without the network's influence.

We provide the simulation code for future research. This tool can help analyze how information spreads and how specific nodes become more central when data is available on the strength of prior beliefs and initial network structures. These simulations may also be useful for forecasting social dynamics, such as identifying potential leaders in a protest or predicting which individuals are most likely to change their minds about participating in an event.

The main contribution of this paper is a theoretical model for collective action that is more general and flexible than traditional global games. By proving the conditions for equilibrium uniqueness, we show that information exchange can be clearly defined and analyzed within our framework. Our comparative statics offer a clear look at how different parameters affect game outcomes, providing intuition on how public trust or social connections shift behavior. Additionally, by incorporating an information exchange network, our model better reflects how players actually gain knowledge about the world. The proposed network formation model helps track how community structures change over time, and our simulation tool provides a practical way to visualize these processes and see how sharing information impacts player decisions.

This paper is organized as follows. The next section introduces the global game model on an information exchange network, defines the equilibrium, and solves the player's maximization problem. We then present comparative statics to examine how network topology and information parameters affect the equilibrium. Following this static analysis, we introduce a second game where players form connections to maximize their expected utility. The penultimate section presents simulations of this dynamic network game, followed by an analysis and interpretation of the results. We conclude with a summary of our findings, their significance, and potential future research directions.

The appendix provides several generalizations of the model. We first present specific network examples and their equilibrium conditions to further illustrate the game. We then analyze variations in the information exchange protocol and player behaviors, followed by different network formation games applied to our framework. Finally, the appendix includes the Python 3 code used for the simulations.

2 The Model

Consider a game with a set of players $\mathcal{I} = \{1, 2, ..., n\}$. Each of them faces a decision whether to take a risky action R or choose a safe option S. Players' payoffs in case of taking the risky action depend on the total share of players choosing R as well as the cost of the action θ. The share of risk-takers is given by k/n, where $k = |\{a_i = R : i \in \mathcal{I}\}|$ is the number of players taking the action R and n is the total number of players in the set $\mathcal{I}$. Payoffs of agent i, with given action profile $a = (a_1, a_2, ..., a_n)$, are given by:

$$u_i(a) = \begin{cases} 0, & \text{if } a_i = S, \\ k/n - \theta, & \text{if } a_i = R. \end{cases}$$

$\theta \in \mathbb{R}$ is the realisation of the model's fundamentals. It is drawn from a distribution $\Theta \sim \mathcal{N}(\bar{\theta}, 1/\alpha)$, which the players know. However, they do not observe θ, only a private, noisy signal $x_i = \theta + \epsilon_i$, where $\epsilon_i \sim \mathcal{N}(0, 1/\beta)$ is i.i.d. normal random variable independent of Θ.

In addition to the private signal, each player observes the signals of each of her neighbors, a subset $\mathcal{N}_i \subseteq \mathcal{I}$ of other players. The players are connected by a network described by a symmetrical matrix $\mathbb{G}$. Players i and j are neighbors if $\mathbb{G}_{i,j} = \mathcal{G}_{j,i} = 1$. The undirected graph $\mathbb{G}$ is common knowledge among players, and $i \in \mathcal{N}_i$ for all agents i. Let $k_i = |\mathcal{N}_i|$ be the size of the neighborhood of player i. For ease of readability, let $y_i = (x_j)_{j \in \mathcal{N}_i}$ denote the information player i receives in the game - the idiosyncratic signal and the neighbors' signals. All communications are truthful.

2.1 Utility maximisation

Based on the prior beliefs and signals they receive, each player obtains a posterior. As both θ and x are normally distributed, it is the average of signals weighted by their accuracy:

$$\hat{\theta}_i := \mathrm{E}_i(\theta|y_i) = \frac{\alpha\bar{\theta} + \beta\sum_{j\in\mathcal{N}_i} x_j}{\alpha + \beta k_i},$$

$$\mathrm{Var}(\theta|y_i) = \frac{1}{\alpha + \beta k_i}.$$

Players also have beliefs regarding other players' posteriors. These are obtained from the common prior and signals the player observes. Therefore, a signal x_j obtained by player i from player j impacts i's own posterior and also their expectations regarding all the neighbors of j. Player i's expectations about others' posteriors are as follows:

$$\mathrm{E}_i(\hat{\theta}_j|y_i) = \frac{\alpha\bar{\theta} + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i)}{\alpha + \beta k_j},$$

$$\mathrm{Var}_i(\hat{\theta}_j|y_i) = \frac{\beta|\mathcal{N}_j \setminus \mathcal{N}_i|}{\alpha + \beta k_j}.$$

The expected payoff of the player depends on the action and the expected cost $\mathrm{E}_i(\theta|y_i)$. Players' expectations about others' posteriors take into account the knowledge they get from their common neighbors $l \in \mathcal{N}_i \cap \mathcal{N}_j$. At the same time, the variance of those expectations decreases as players have more common neighbors. An edge with any player j impacts expectations on all of their connections. Adding a new connection has a nonlinear effect, depending on its centrality in the network. As we will see in the following chapter on endogenous networks and the examples in the appendix, it has a strong impact on the game's outcome.

The action is monotonic in the expected cost. The higher it is, the lower the action the player would choose, and the differences are decreasing. From Topkis's theorem (Topkis, 1978) we see that without loss of generality, arg max of the payoff is a decreasing function of $\mathrm{E}_i(\theta|y_i)$. Therefore, each player has a threshold level $T_i \in \mathbb{R}$ of the expected cost of taking the risky action, above which she would choose to stay safe $a_i = S$ if $\hat{\theta}_i > T_i$.

$$a_i(\hat{\theta}_i) = \begin{cases} R & \text{if } \hat{\theta}_i \leq T_i, \\ S & \text{if } \hat{\theta}_i > T_i. \end{cases} \tag{1}$$

Let $V(T_{-i}|y_i)$ be player i's expected payoff of taking a risky action given the signals obtained from the network and the thresholds of others. As the threshold is strictly connected to the expected payoff,

$$a_i(\hat{\theta}_i) = R \iff V_i(T_{-i}|y_i) \geq 0,$$

and

$$\hat{\theta}_i = T_i \iff V_i(T_{-i}|y_i) = 0.$$

The expected utility of risky action $V(T_{-i}|y_i)$ is given by the expected θ and expectations about the beliefs of other players:

$$V(T_{-i}|y_i) = \frac{1 + \sum_{j=1}^{n} Pr(\hat{\theta}_j \leq T_j|y_i)}{n} - \hat{\theta}_i.$$

Therefore:

$$T_i = \frac{1 + \sum_{j=1}^{n} Pr(\hat{\theta}_j \leq T_j|y_i)}{n}.$$

In this game, the strategy is defined by a vector of thresholds $T = (T_1, T_2, ..., t_n)$. T_i is the best response to expectations of T_{-i} given player's knowledge of the world. Therefore, T_i would be equal to the sum of $1/n$, which is player i's contribution to the payoff from the game if she takes the action, and the share of players expected to take the action:

$$T^*(y_i) = \frac{1 + \sum_{j=1}^{n} \left(1 - \Phi\left(\frac{\mathrm{E}_i(T_j) - \mathrm{E}_i(\hat{\theta}_j|y_i)}{\mathrm{Var}(\hat{\theta}_j|\hat{\theta}_i)^{1/2}}\right)\right)}{n}. \tag{2}$$

Players' path of deciding on their action is following. First, they observe their and their neighbors' private signals (y_i) and build their own posterior ($\hat{\theta}_i$). Second, they form expectations about others' posteriors ($\mathrm{E_i}(\hat{\theta}_j|y_i)$). Third, they form expectations about others' thresholds ($\mathrm{E_i}(T_j)$). Fourth, they combine those two to obtain own threshold (T_i). Finally, they take the decision by comparing their posterior to their threshold (2.1).

Each player's equilibrium threshold is given by a closed-form equation above 2.1, which has a unique value if players' expectations about others' thresholds. This gives us the definition of our perfect Bayesian equilibrium of this game.

DEFINITION 1. *An equilibrium of this game is a vector of thresholds $T^* = (T_1^*, T_2^*, ..., T_n^*)$, such that for every player i, T_i^* is the best response of player i to the expected thresholds of the other players $\mathrm{E}_i(T_{-i})$ established based on the knowledge player i obtained from the network.*

Players get expectations of others' thresholds by applying the same formula to a vector of initial expectations. Then, players convert a vector of all expected thresholds and get a final one based on player i's knowledge of the world. For each player j, the formula is equivalent to the one for player i's threshold 2.1:

$$E_i(T_j) = \frac{1 + \sum_{k=1}^{n} \left(1 - \Phi\left(\frac{\mathrm{E}_i(T_k) - \mathrm{E}_i(\hat{\theta}_k|y_i)}{\mathrm{Var}(\hat{\theta}_k|\hat{\theta}_k)^{1/2}}\right)\right)}{n}.$$

The formula for the equilibrium threshold 2.1 has a closed form. Therefore, if the expected thresholds are uniquely valued for each player, the game would have a unique equilibrium T^*.

THEOREM 1. *A vector T is the unique equilibrium of the game if for any players i and j: $\frac{\alpha}{\beta} < 2\pi|\mathcal{N}_j \setminus \mathcal{N}_i|n^2 - k_j$.*

Proof. Let $T = (T_1, T_2, ..., t_n)$ be a vector of thresholds that are best responses to the thresholds of all the other players. To prove the uniqueness of the equilibrium in this game, we need to show that $BR(T_{-i})$ is a contraction. As $f : R_n \mapsto R_n$ from the Banach fixed-point theorem, we need to show that the average marginal impact of an increase in T_j on T_i is between 0 and 1. To show that we take the derivative with respect to T. Therefore, the condition for the uniqueness of equilibrium is:

$$|\frac{\partial T^*}{\partial T_j}(T_{-i}, y_i)| < 1,$$

$$\frac{\partial T^*}{\partial T_j}(T_{-i}, y_i) = -\frac{1}{n}\text{Var}(\hat{\theta}_j|\hat{\theta}_i)^{-1/2}\phi\left(\frac{T_j - \text{E}_i(\hat{\theta}_j)}{\text{Var}(\hat{\theta}_j|\hat{\theta}_i)^{1/2}}\right) = -\frac{1}{n}\left(\frac{\alpha + k_j\beta}{2\pi\beta|\mathcal{N}_j \setminus \mathcal{N}_i|}\right)^{1/2} \underbrace{e^{-\frac{\left(\frac{T_j - \text{E}_i(\hat{\theta}_j)}{\text{Var}(\hat{\theta}_j|\hat{\theta}_i)^{1/2}}\right)^2}{2}}}_{\leq 1}.$$

Therefore:

$$\frac{\partial T^*}{\partial T_j}(T_{-i}, y_i) < 1 \iff \left(\frac{\alpha + k_j\beta}{2\pi\beta|\mathcal{N}_j \setminus \mathcal{N}_i|}\right)^{1/2} < n \iff \frac{\alpha}{\beta} < 2\pi|\mathcal{N}_j \setminus \mathcal{N}_i|n^2 - k_j.$$

If this condition is met, player i has a unique expectation about player j's threshold. If this is true for all other players, player i will have a unique equilibrium threshold T_i^*. If it holds for all players, the vector T^* will be unique. □

This result shows that the unique equilibrium exists only if there is enough noise. This condition can be met with a sufficiently large society (n) if the network is not connected. If either player is certain about the common prior and does not update their beliefs, or they have too many signals about other members of the network, the uncertainty becomes too small. They will again have multiple equilibria.

Such intuition goes according to results by Morris and Shin (2000) and Hellwig (2002). If there is too much certainty, the game becomes a classic game of complementarities with multiple equilibria. Following Morris and Shin, if $\frac{\alpha+k_j\beta}{2\pi\beta|\mathcal{N}_j\setminus\mathcal{N}_i|}$ goes to zero, all players' thresholds become equal to 1/2.

We analyze the game in ex-ante and interim scenarios, examining the threshold equilibrium. Such an approach also allows us to analyze dynamic versions of such games, where players' thresholds change as the network develops.

We chose the perfect Bayesian equilibrium, as defined above, and find the optimum threshold analytically. In a situation where $\beta \to \infty$ and players observe the actual state of the world, we can follow the iterative approach of Frankel et al., 2003. This approach would yield the same results. We analyze our condition in the limit in Corollary 1.

COROLLARY 1. *As the variance of players' private signals becomes zero or $\beta \to \infty$, the necessary condition to achieve a unique equilibrium becomes $2\pi > \frac{k_j}{|\mathcal{N}_j \setminus \mathcal{N}_i| n^2}$ for average j in i's neighborhood.*

The corollary follows strictly from the Theorem and that $lim_{\beta \to \infty} \frac{\alpha}{\beta} = 0$. It shows that, in the limit, the connectedness of the network is a crucial aspect of achieving a unique equilibrium, as proposed in Theorem 1.

Corollary 1 highlights that our model is in line with the main findings from the literature on the global games' equilibria (Leister et al., 2021; Morris and Shin, 2000). In the next parts of the paper, however, we focus on the threshold equilibrium. As we operate on the limit $\beta \to \infty$, information exchange becomes meaningless. Each player obtains the same information, and they have no reason to learn about others' beliefs. The perfect Bayesian equilibrium described in Theorem 1 is the main finding of the model and can be the base for theoretical, verifiable predictions for simulations and potential future research.

2.2 Equilibrium Comparative Statics

Having proved the existence of a unique equilibrium, we can analyze the impact of changes in the parameters of the model and the state of the world on the outcomes of the game. We start by analyzing the impact of the observed information and its variances on players' thresholds. Then, we focus on the network and society.

COROLLARY 2. *Observing a higher private signal x_i leads to a lower equilibrium threshold T_i.*

Proof. We are showing that function T^* of y_i that returns player's optimal threshold T_i is decreasing in x_i or that it's derivative with respect to x_i is lower than 0. The player's private signal does not impact the threshold directly, but influences the player's expectation about the state of the world and expectation about player i's neighbors, who use it as one of their signals.

$$\frac{\partial T^*}{\partial x_i}(y_i) = \frac{1}{n}\sum_{j=1}^{n} -\frac{\beta}{\alpha+\beta k_j}\mathrm{Var}_i(\hat{\theta}_j)^{1/2}\phi\left(\frac{\mathrm{E}_i(T_j)-\mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right),$$

which is always lower than 0. Therefore, the best response decreases in x_i. As the best response, and the function T^* is monotone in T_{-i}, as per Theorem 1, our equilibrium will decrease as well from Tarski's fixed-point theorem (Tarski, 1955). The higher the player's private signal, the lower the equilibrium threshold.

$$\frac{\partial^2 T^*}{\partial x_i^2}(y_i) = \frac{1}{n}\sum_{j=1}^{n} -\left(\frac{\beta}{\alpha+\beta k_j}\right)^2\mathrm{Var}_i(\hat{\theta}_j)\phi'\left(\frac{\mathrm{E}_i(T_j)-\mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right).$$

The first part of the right-hand side of the equation is always less than 0. However, the derivative of the normal probability distribution function is less than 0 for values greater than 0. Therefore, if player i believes $T_j > \mathrm{E}_i(\hat{\theta}_j)$, the whole formula would be greater than 0. If this weighted average for all $j \neq i$ is greater than 0, the effect of increasing the private signal is diminishing. □

The intuition behind the corollary is that a player's private signal has a magnified effect on her probability of taking the risky action. One part of it comes directly from the change of the player's posterior and expectations about others' posteriors. It is worth reminding that obtaining a signal from another player j not only influences one's own beliefs but also gives an indication about the posteriors of j's neighbors.

The other part of the effect is the change in threshold. The less beneficial the private signal, the stricter the threshold becomes. This can be proven by:

$$\frac{\partial T_i^*}{\partial x_i}(y_i) = \frac{1}{n}\sum_{j=1}^{n} -\frac{\beta}{\alpha+\beta k_j}\mathrm{Var}_i(\hat{\theta}_j)^{1/2}\phi\left(\frac{\mathrm{E}_i(T_j)-\mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right).$$

COROLLARY 3. *As the precision of the prior-α increases, player i's threshold T_i will increase if the common prior, $\bar{\theta}$ is lower than player i's expectations of others' posteriors and their thresholds.*

Proof. We want to check if an increase in α increases the player's threshold T_i. We analyze the derivative of the best response function wrt. α and check what the sign of the given derivative is.

$$\frac{\partial T^*}{\partial \alpha}(y_i) = \frac{1}{n}\sum_{j=1}^{n} \frac{\frac{1}{2}(\mathrm{E}_i(T_j)(\alpha + k_j\beta) + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i)) - \bar{\theta}(\frac{1}{2}\alpha + k_j\beta)}{(\alpha + k_j\beta)^{3/2}} \phi\left(\frac{\mathrm{E}_i(T_j) - \mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right).$$

This gives us a sufficient condition for the derivative to be positive.

$$\frac{\partial T^*}{\partial \alpha}(y_i) > 0 \Leftarrow \forall j \frac{1}{2}(\mathrm{E}_i(T_j)(\alpha + k_j\beta) + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i)) > \bar{\theta}(\frac{1}{2}\alpha + k_j\beta).$$

The right-hand side of that implication simplifies to

$$\frac{\alpha\bar{\theta} + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i)}{\alpha + k_j\beta} > 2\bar{\theta} - \mathrm{E}_i(T_j).$$

Therefore the final sufficient condition for the derivative of $T*$ with respect to α can be written as: $\forall j \frac{\mathrm{E}_i(\hat{\theta}_j|y_i)+\mathrm{E}_i(T_j)}{2} > \bar{\theta} \implies \frac{\partial T^*}{\partial \alpha}(T_{-i}, y_i) > 0$.

Following the reasoning from the proof of Corollary 2, we know that this implies that the equilibrium will increase. Therefore, if the condition above is met, the increase in α will lead to a higher equilibrium threshold. □

The intuition behind this corollary is that if the prior is lower, indicating a lower cost of taking the risky action, players will be more inclined to take the risky action if this signal is more accurate. Otherwise, if they observe higher accuracy on a signal indicating higher cost of taking the risky action, their posterior will increase, and they will be less likely to take the aforementioned risky action.

COROLLARY 4. *As the precision of private signal - β gets higher, player i's threshold T_i will increase if the signals obtained by players are lower than the prior $\bar{\theta}$.*

Proof. Similarly to 3, we show the impact of an increase in β on the best response function.

$$\frac{\partial T^*}{\partial \beta}(y_i) = \sum_{j=1}^{n} -\frac{\alpha|\mathcal{N}_j \setminus \mathcal{N}_i|(\mathrm{E}_i(T_j)(\alpha+\beta k_j) - \bar{\theta}(\alpha+\beta k_j) + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l - |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i))}{2n(\alpha+\beta k_j)^3 \mathrm{Var}_i(\hat{\theta}_j)^{3/2}} \phi\left(\frac{\mathrm{E}_i(T_j) - \mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right).$$

Following the steps of 3 we find the sufficient condition under which the derivative above is positive. It can be stated as

$$\frac{\partial T^*}{\partial \beta}(T_{-i}, y_i) > 0 \Leftarrow \forall j - \frac{\alpha|\mathcal{N}_j \setminus \mathcal{N}_i|(T_j(\alpha+\beta k_j) - \bar{\theta}(\alpha+\beta k_j) + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l - |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i))}{2n(\alpha+\beta k_j)^3 \mathrm{Var}_i(\hat{\theta}_j)^{3/2}} > 0.$$

Further simplification leads to a form similar to that of 3.

$$\bar{\theta} > \frac{\mathrm{E}_i(\hat{\theta}_j|y_i) + T_j}{\left(1 + \frac{\alpha}{\alpha+\beta k_j}\right)} \implies \frac{\partial T^*}{\partial \beta}(T_{-i}, y_i) > 0$$

□

In accordance with the previous corollary, this one indicates that as the accuracy of private signals increases, players will increase their threshold if the private signals they obtain suggest low costs of taking the risky action.

COROLLARY 5. *Adding an edge to the network - a connection between player j and some other player, but not i - will increase player i's threshold if the common knowledge of player i and j (weighted average of prior and shared private signals) is higher than* $\mathrm{E}_i(T_j)$.

Proof. We want to check the effect of increasing the connectedness of the network. We take the derivative of $T^*(y_i)$ with respect to k_j to see how player i's threshold would react to a change in the size of a neighborhood of other player j.

$$\frac{\partial T^*}{\partial k_j}(y_i) = \frac{\alpha\bar{\theta} + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\mathrm{E}_i(T_j)) - \mathrm{E}_i(T_j)(\alpha+\beta k_j)}{\beta|\mathcal{N}_j \setminus \mathcal{N}_i|^2 n} \phi\left(\frac{\mathrm{E}_i(T_j) - \mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)}\right).$$

As $\phi(...)$ is always greater than 0, we are left with a condition for the derivative to be always greater than 0:

$$\frac{\partial T^*}{\partial k_j}(y_i) > 0 \iff \frac{\alpha\bar{\theta} + \beta(\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\mathrm{E}_i(T_j))}{\alpha + \beta k_j} > \mathrm{E}_i(T_j)$$

. After removing T_j from the left-hand side of the inequality, we are left with a formula for the shared knowledge posterior between i and j.

$$\frac{\partial T^*}{\partial k_j}(y_i) > 0 \iff \frac{\alpha\bar{\theta} + \beta\sum_{l\in\mathcal{N}_i\cap\mathcal{N}_j} x_l}{\alpha + \beta|\mathcal{N}_i \cap \mathcal{N}_j|} > \mathrm{E}_i(T_j).$$

□

The results of Corollary 5 indicate that if player i sees that j is likely to have a high prior $\hat{\theta}_j$, according to the shared information they got, they'll perceive new edges as a source of potentially lower private signals and a decrease in j's posterior. As j is more likely to join the action, the expected utility from joining increases, as does their T_i.

Intuitively, it may be thought that players would prefer to get as much information as possible. To verify that we analyze the expected shift in strategy from adding a player "j" to i's network. The effect of such addition on i's threshold is positive, leading to higher expected utility from taking the risky action, if and only if $\mathrm{E}_i(T_j) < x_i$. It indicates that players will be only willing to extend their network, not to gain additional information, but to influence others to take the risky action.

We can see that players with high private observation of costs x_i will not find it beneficial to extend their network. A player's utility never increases as others' thresholds decrease. Therefore, if a player thinks others will take the risky action, they will not want to share their high private signal.

These findings indicate why the "echo chambers" are created. If players have a relative ease of establishment of new edges, only those who are convinced that the risky action is an optimal decision would exchange information. All the others would get no additional value from participating in information sharing.

2.3 Outside of the condition for the uniqueness of the equilibrium

Let's examine, what the outcomes of the game would be if the condition of the Theorem 1 is not met. The left-hand side of the condition from Theorem 1 states that there will be a unique equilibrium if α is not too large compared to β. The right-hand side indicates that players cannot have too many common connections. Otherwise, the uncertainty would fade, with players able to estimate each other's posteriors accurately. Theorem 1 suggests that the vector of thresholds T would be unique if those conditions are met by all n players. However, we may observe a situation where only a portion does or does not meet the condition.

Consider a network with n players that contains a clique of m players and $n-m$ players having no connections. Every member of the clique is connected with all other members. For them, the condition for uniqueness can never be met as the right-hand side of the inequality is less than 0. For every other player, they will have a unique threshold if $\frac{\alpha}{\beta} < 2\pi n^2 - 1$.

If the condition is met, we may use comparative statics for individual players of the $n - m$ not in the clique, even if the condition does not hold for all, and the whole game does not have a unique equilibrium. The only situation where nothing can be said about any player's choice is when neither meets the condition. In that case, the game is too close to the coordination game for each player, and no player has a unique equilibrium. In that case, the game should be analyzed in the same way as the coordination games are.

A more detailed analysis of what happens in cases of high connectivity in the network can be found in Appendix A. As we analyze the complete network or a star network, we find situations where none or some of the players would not meet the condition for uniqueness.

3 Endogenous Network

Now we analyze the same game in a setting where, before deciding on which action to take, players may establish new edges in the network. We want to observe how the connection structure evolves when players try to maximize their expected utility by getting more information and sharing their views on the network. To do that, we add one more step to the game. Now, after observing their own and their neighbors' private signals, players may choose to establish costly links.

Connections are created randomly. Between two individuals that choose to do so, graph edges are created at random, similarly to (Acemoglu and Jackson, 2017). It does not follow the network formation model as in (Barabási and Albert, 1999), where the new current network influences new nodes. The proposed approach aims at modeling social interactions of people interested in specific subjects, like joining a protest or a movement, or friendship networks which tend to be more random (Jackson and Rogers, 2007). We do not analyze the case of dropping edges, as in our setting, players would never have an incentive to take such action.

Two other ways of establishing connections are proposed in the appendix. Namely, we analyze models where 1) players can select their new neighbors (similarly to (Myerson, 1977)), 2) the so-called "information rooms" where players can choose to connect to everyone in a clique. The results follow the same intuition but change in the magnitude of observed effects.

The main game's timeline looks as follows:

- 0 θ and private signals are drawn by nature
- 1 Players update their posteriors
- 2 Players decide whether they want to connect with others
- 3 Pairs drawn at random between unconnected players who decided to create edges
- 4 Edges are created at a cost c and private signals exchanged
- 5 Steps 1-4 repeated until there are no 2 unconnected players who want to create edges
- 6 Players decide what decision to take

Establishing an edge is costly for players. The cost c is common for every person who is being connected by a new edge. A player would only choose to establish a node if the expected increase in utility was higher than the cost c. In expectation, the player's information gained by establishing a node would be $\mathrm{E}_i(x_j) = \hat{\theta}_i$. Players believe that their posterior is correct and therefore only expect to confirm their knowledge. Gains from exchanging information come from the expected increase in their new neighbors' beliefs. The increased probability of taking risky actions by other players causes a higher expected payoff for player i.

Points 2-4 show that only players who are interested in forming a network can get paired. In our model, an individual who does not consider information exchange beneficial will never form new edges and obtain new signals. At the same time, players do not cut ties with each other. Keeping existing nodes is cost-free, and players would not benefit from losing signals from their neighbors.

The structure of the game ensures that an equilibrium always exists and is reached iteratively. Iterations of the network formation model ensure that until at least 2 unconnected players are willing to establish new edges, the game will proceed. The randomness of the process leads to a multiplicity of equilibria. Depending on the starting point, the game will reach a different equilibrium.

THEOREM 2. *Player i will only be willing to establish a new edge if they are willing to take the risky action and their private signal x_i is lower than the average expected posterior of others in the society.*

Proof. First, only players who want to take risky action will be interested in establishing new edges. If they are not taking the action, their ex-ante utility is 0, and in their expectation, it won't change as they obtain new information. Therefore, the only effect of establishing a new edge is bearing the cost c.

As the only players creating new edges will be the ones already convinced to take the risky action, their goal will be to maximize their expected utility from the action. Therefore, their goal is to maximize the aggregated probability of joining the action by other players.

$$Pr(a_j = 1) = Pr(\hat{\theta}_j \leq T_j) = \Phi\left(\frac{T_j - \mathrm{E}_i(\hat{\theta}_j)}{\mathrm{Var}_i(\hat{\theta}_j)^{1/2}}\right).$$

The variance of i's expectation regarding j's posterior always decreases in the effect of establishing an edge between them.

$$\frac{\partial \mathrm{Var}_i(\hat{\theta}_j)}{\partial j \in \mathcal{N}_i} = -\beta \frac{\alpha + \beta(k_j + |\mathcal{N}_j \setminus \mathcal{N}_i|)}{(\alpha + \beta k_j)^2}.$$

Therefore, adding a new edge between player i and player j will always increase i's utility if j has $\bar{\theta}_j \leq T_j$ and its neighbors k have $\bar{\theta}_k \leq T_k$.

Player i would only be willing to establish a new connection with j if an expected increase in the probability that a new connection j from having that edge would grow by more than c. Therefore establishing a new edge benefits player i if it decreases $\hat{\theta}_j$ or, in some cases by decreasing $\mathrm{Var}_i(\hat{\theta}_j)$.

$$Pr(a_j = 1 | j \in \mathcal{N}_i) - Pr(a_j = 1 | j \notin \mathcal{N}_i) =$$

$$\frac{e^{-\left(\frac{T_j - \mathrm{E}_i(\hat{\theta}_j)}{(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2}}\right)^2}}{\pi^{1/2}} \Bigg(\big(T_j - \mathrm{E}_i(\hat{\theta}_j)\big) \frac{\alpha + \beta(k_j + |\mathcal{N}_j \setminus \mathcal{N}_i|}{2|\mathcal{N}_j \setminus \mathcal{N}_i|(\alpha + \beta k_j)(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2})}$$

$$- \big(x_i(\alpha + \beta k_j) - (\alpha\bar{\theta} + \beta(\sum_{l \in \mathcal{N}_i \cap \mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i | \hat{\theta}_i))\big) \frac{(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2}}{2|\mathcal{N}_j \setminus \mathcal{N}_i|(\alpha + \beta k_j)} \Bigg).$$

If the equation above is greater than 0 for average j, the player will be willing to establish new edges. We can see that the increase in player i's utility will be bigger the higher the $T_j - \mathrm{E}_i(\hat{\theta}_j)$ and $\mathrm{E}_i(\hat{\theta}_j) - x_i$. □

Let us analyze the intuition behind this theorem. First, the higher the current expectation of j's probability of taking the risky action, the more utility i will get from the lower variance of this expectation. Second, the more we increase the chances of j taking the risky action, the better. To make it work, players who see if the average connection would improve their expected utility decide to form new edges. This follows findings by Sinapuelas and Ho, who show that people with more connections in social media are more inclined to share their opinions and influence others (Sinapuelas and Ho, 2019).

Under such conditions, only players whose posterior $\hat{\theta}_i$ is low will decide to establish new edges. Players with low x_i have lower posterior, therefore lower expectations of others' posteriors. That would cause only players who are already willing to take risky actions to establish new edges and confirm their beliefs. The game would result in very dense connections between players with low posteriors and few connections for those who observed high private signals.

The random edge assignment between players who wish to expand their network follows a uniform distribution. It is similar to the network formation model proposed by (Jackson and Rogers, 2007). The only difference from their paper is that in our context, players decide to participate in the edges draw before it happens. Jackson and Rodgers first draw edges and then check if nodes are willing to establish those.

3.1 Simulation

Endogenous network games can be simulated. The Python code from Appendix D allows for selecting parameters of the game (α, β, n, and $\bar{\theta}$). Then we can either analyze the game with a hand-made network, generate an Erdős-Rényi (ER), Barabási-Albert (BA) graph (Barabási, Albert (1999), Erdős, Rényi (1959)) or have an endogenous network as in a game proposed above.

To simulate the endogenous network formation in the game above, we run it 100 times at different levels of $\bar{\theta}$ from 0 to 0.9 with $\alpha = \beta = 4$ and c = 0.001. We compare the number of players who decide to take risky action in a game without any information sharing and after forming a network. Later on, we check how the number of connections differs between players that initially obtained a private signal smaller than $\bar{\theta}$ and those that observed a higher realization of fundamentals.

The expectation regarding the first test is that the number of players deciding to take a risky action should remain unchanged. Only players with low expected theta would decide to exchange information. Following that, the degree of players observing small private signals should be substantially higher than that of the ones with a higher posterior.

Results confirm the intuition. Players with lower private signals establish more nodes at each level of fundamentals. However, if the state of the world is expected to provide positive utility for risk-takers, all players decide to establish more nodes to confirm their views. This is represented by the β-index, a measure of the connectedness of a network, defined as a ratio of edges to nodes in a graph. It was chosen as the simplest way to show the diminishing number of connections with less favorable states of the world. At $\bar{\theta} = 0$ player is very certain about the positive outcome of taking a risky action and is establishing new edges to create a highly connected network. With $\bar{\theta} = 0.9$, there are barely any edges in the graph, and all of them are between players with the lowest private information.

Decisions taken by players are influenced by the network. At low levels of fundamentals, initially, almost everyone decides to take a risky action. At $\bar{\theta} < 0.5$, more players decide not to play after establishing connections. If the realization of θ is higher than what is expected by players based on $\bar{\theta}$, they will be willing to play at first, but will change their mind after learning from their new neighbors. The opposite happens at higher levels $\bar{\theta}$. There, players initially are not willing to take the risk, as their prior is that the theta is too high, but they may learn from the network that the state drawn by nature is more beneficial.

In the tables below, we present simulation results for each $\bar{\theta}$ in a different row. We look at the share of players that, on average, would decide to take a risky action if no network were present (initial risk takers %). The third column presents a similar share, but after an appropriate network was established (risk-takers % with the network). The fourth column shows the average number of degrees at the end of the game for players with private signals lower than prior - the ones that initially upgrade their posterior towards lower costs of taking the risky action (avg. degree with x $\leq \bar{\theta}$). The next number is the opposite, the average degrees of players who observed a higher private signal (avg. degree with x $> \bar{\theta}$). The penultimate column presents the average β index of the network, which measures its connectivity with a simple formula $\beta = \frac{E}{V}$, where E is the number of edges and V - vertices (β index). In the last column, we show the share of networks that are connected (% connected graphs).

3.1.1 Random Connections

Results with c = 0.001 without initial network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	91.20	5.96	5.88	2.33	73.0
0.1	95.30	82.62	5.32	4.90	2.16	51.5
0.2	93.55	82.15	5.03	4.58	2.08	44.0
0.3	89.77	80.73	4.88	4.17	1.87	20.5
0.4	70.32	63.53	4.43	3.21	1.68	9.5
0.5	52.83	51.16	3.69	2.30	1.48	5.5
0.6	41.84	40.62	2.97	1.74	1.00	1.0
0.7	20.38	22.97	2.02	1.08	0.73	0.0
0.8	9.60	10.05	1.52	1.00	0.58	0.0
0.9	3.18	1.93	1.01	1.00	0.51	0.0

The table above shows that the introduction of the network decreases aggregate utility across players. With low values of $\bar{\theta}$, where players taking a risky action get a high utility, the network causes more players to choose a safe option, decreasing their utility. At higher levels of cost of taking the action, more players would do so. In both cases, the introduction of a network creates an environment where extreme signals about the costs are shared and cause a higher share of the population to take an action that, ex-post, provides them with lower utility. The effect does not appear at $\bar{\theta} = 0.9$ or higher, as no players are willing to exchange information at this level of cost.

To check the robustness of these results, we compare them with games with initial ER and BA networks. The intuition is that having some connection before obtaining a private signal and exchanging information first would cause less variance in posteriors between players. The results should follow the same direction as in the case of the empty graph. In states of the world with low theta, fewer players choose to take risky action. In the ones with high cost, more. It is caused by players who would normally see no need to participate in the game, having an initial impact.

Table 2. Results with c = 0.001 with initial Erdős-Rényi network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	91.40	5.16	4.91	2.48	92.0
0.1	95.30	86.64	5.31	4.87	2.46	84.0
0.2	93.55	81.60	4.95	4.40	2.26	72.0
0.3	89.77	76.00	4.65	3.76	2.02	53.0
0.4	70.32	70.74	4.65	3.29	1.89	38.0
0.5	52.83	57.23	3.96	2.39	1.57	18.0
0.6	41.84	39.98	2.94	1.61	1.24	12.0
0.7	20.38	22.76	2.10	1.16	0.94	3.0
0.8	9.60	13.36	1.45	1.03	0.72	2.0
0.9	3.18	2.09	1.03	1.01	0.52	0.0

With the initial Erdős-Rényi network, players are initially connected and they have a lower propensity to exchange information. However, the same effect as in Table 1 is noticeable. The introduction of endogenous networks makes more radical players' views more shared and causes more players to take risky action when it is more costly.

The effect is most visible in the Barabási-Albert network. The central nodes will have a great impact on the posteriors of others. This can be observed as a nonlinearity of average degrees and β index with θ. If central players have a much lower private signal, they will cause more players than usual to search for additional information.

Table 3. Results with c = 0.001 with initial Barabási-Albert network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	91.05	5.19	5.02	2.53	92.0
0.1	95.30	87.98	5.12	4.84	2.45	85.0
0.2	93.55	84.14	4.96	4.33	2.25	69.0
0.3	89.77	76.67	4.71	3.70	2.03	53.0
0.4	70.32	72.96	4.35	3.30	1.88	41.0
0.5	52.83	52.42	3.85	2.36	1.51	20.0
0.6	41.84	42.34	3.00	1.56	1.25	7.0
0.7	20.38	16.74	2.01	1.22	0.90	2.0
0.8	9.60	14.91	1.79	1.25	0.86	5.0
0.9	3.18	3.36	1.42	1.27	0.68	1.0

In the Barabási-Albert network, which is the most resembling of the social networks observed in the empirical data, the effects are the strongest. Even in cases which provide less utility to players, such as $\bar{\theta} > 0.8$, a player with a low private signal x may be the most central node of the network and influence multiple other players. It creates echo chambers, leading to the lowest ex-post aggregate utility levels.

Adding the cost of establishing edges changes the outcomes of the network formation. Fewer players decide to form connections, as the expected gain in utility does not cover the cost. At $\bar{\theta} > 0.8$, no players decide to form connections. As the network is less connected, the effects of information exchange on the decision taken are smaller, and the share of players deciding to take a risky action before and after creating a network is more similar. At $c = 0.05$, no edges are being established at any level of prior.

3.2 Simulation results

What is noticeable in the results of simulations is that with the introduction of the network, fewer players will decide to take the risky action with $\bar{\theta} < 0.5$, or with low cost. Whereas with $\bar{\theta} > 0.5$, when the costs of taking the action are high, the participation in the action will be, on average, higher than without the network.

This reflects the "social bubble" intuition often mentioned in the political science papers (Allcott and Gentzkow, 2017). Only players observing low private signal despite high $\bar{\theta}$, who we can compare to people having positive views about participating in unpopular events, will be willing to join the information sharing about that case. They will form a clique or a very densely connected group. If their network contained players with less extreme views, they are likely to update their prior enough to join the action themselves. This way, imposed networks cause players to take actions that ex-post would decrease their utility.

On the other hand, if $\bar{\theta}$ is low and the expected costs of participation in the action are low, players may find themselves in a neighborhood of others who observed higher costs. A player like that may be influenced not to make a decision that would increase their utility. In our model, players who observe high θ_i will not create a social bubble, as they will never consider taking the action and have no expected utility gain from information exchange. They may, however, convince their neighbors not to take the action. That would decrease their ex-post utility.

In a model where players only consider taking a binary action, having a network that evolves around information on the subject of that action, society will, on average, get less ex-post utility. It is caused by the fact that only players who are convinced to participate would be willing to take the extra cost to convince others to do so as well. They'll form a dense "echo chamber" that would enhance the views of the members of the group.

4 Concluding remarks

We can draw two main conclusions from this paper. First of all, it is possible to analyze the equilibria in games of collective decision-making where we do not make assumptions of either full knowledge of others' expectations or none at all. Given that players have enough uncertainty in the common prior, we show that there exists a unique equilibrium of a global game on a network. This is an important step in achieving a model most resembling a process of taking collective action in society.

We also show how to model the network endogeneity in such a framework. We propose several approaches to forming such a network, achieving intuitive results in all of them. Our model can be used to model social interactions in different situations, like direct connections or joining groups on social media. This can help describe the effects of people looking for new information from various sources and used as a theoretical background for future sociological research.

The second main conclusion is that such games can be easily simulated and analyzed, giving intuitive results. This indicates that research on the estimation of game parameters can be done by having a network structure and an outcome of such a game. This will allow us to be able to measure the information exchange processes in societies. Our simulations show various possible scenarios and the effect the network has on the outcomes of the game. An important theoretical prediction is that an endogenous network, evolving around information sharing on a binary-choice action, on average decreases the utility of players in the society.

As the model is easily generalized and with minor adjustments and without changing the main results can be adapted to various situations that may arise in modeling social interactions. Building on the input of this paper, we can move on to econometric modeling of the scale of communication between members of society. The findings of the paper may also lead to economic experiments on the diffusion of ideas on networks.

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Appendix A Examples of static networks

In the following section, we analyze several examples of networks in the framework of our model. We show what the conditions are for the uniqueness of the equilibrium. In several cases, we analyze what happens when the condition is not met.

A.1 Null Network

The case of a network where every player observes just the common belief and private signal. This situation is equivalent to the one presented by Morris and Shin (2000). As players have the same beliefs as everyone else in the game, they will have a common threshold. Each player will look at the probability that others have a lower posterior about the costs than her.

$$\hat{\theta}_j \leq \hat{\theta}_i \iff \frac{\alpha\bar{\theta} + \beta x_j}{\alpha + \beta} \leq \hat{\theta}_i \iff x_j \leq \hat{\theta}_i + \frac{\alpha}{\beta}(\hat{\theta}_i - \bar{\theta}),$$

$$Pr(\hat{\theta}_j \leq \hat{\theta}_i | \hat{\theta}_i) = Pr(x_j \leq \hat{\theta}_i + \frac{\alpha}{\beta}(\hat{\theta}_i - \bar{\theta})) = \Phi\left(\frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)}\right)^{1/2}(\hat{\theta}_i - \bar{\theta}).$$

From this equation, we can get the common threshold:

$$T = \Phi\left(\frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)}\right)^{1/2}(T - \bar{\theta})),$$

this gives us a condition for the uniqueness of T:

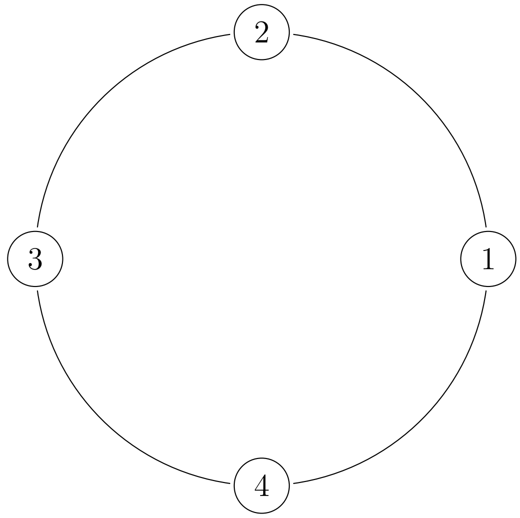
$$\frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)} \leq 2\pi.$$

And we know from Morris and Shin, 2001 that as $\frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)}$ goes to 0, T goes to 1/2.

A.2 Complete Network

A complete network implies that every player has the same posterior about θ. Therefore, the game becomes one of complete information and a coordination game. This type of game has multiple equilibria as shown in multiple papers, with the best-known ones by Diamond and Dybvig (1983) and Obstfeld (1986).

A.3 Cycle Network



In a cycle network, we talk about a symmetrical situation, where each player has exactly 2 neighbors. Then:

$$\hat{\theta}_1 = \frac{\alpha\bar{\theta} + \beta(x_1 + x_2 + x_4)}{\alpha + 3\beta},$$

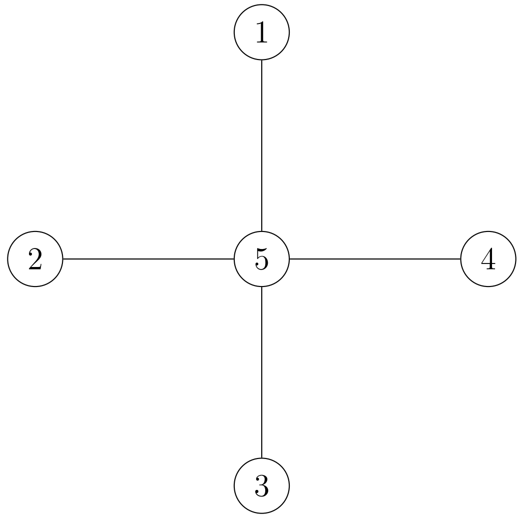
$$\mathrm{E}_1(\hat{\theta}_2) = \frac{\alpha\bar{\theta} + \beta(x_1 + x_2 + \hat{\theta}_1)}{\alpha + 3\beta}.$$

And the equilibrium threshold is unique iff:

$$\frac{\alpha}{\beta} < 32\pi - 3.$$

If the condition is not met, then for each player, we have two equilibrium choices. The game then becomes equivalent to the one described in the "Complete Network" section, despite some uncertainty about the non-neighbors.

A.4 Star Network



Star network is a situation where each "outside" player is connected only to the one in the middle, who has every member of the network in his neighborhood.

$$\hat{\theta}_1 = \frac{\alpha\bar{\theta} + \beta(x_1 + x_5)}{\alpha + 2\beta},$$

$$\mathrm{E}_1(\hat{\theta}_2) = \frac{\alpha\bar{\theta} + \beta(x_5 + \hat{\theta}_1)}{\alpha + 2\beta},$$

$$\mathrm{E}_1(\hat{\theta}_5) = \frac{\alpha\bar{\theta} + \beta(x_1 + x_5 + 3\hat{\theta}_1)}{\alpha + 5\beta},$$

$$\hat{\theta}_5 = \frac{\alpha\bar{\theta} + \beta(x_1 + x_2 + x_3 + x_4 + x_5)}{\alpha + 5\beta}$$

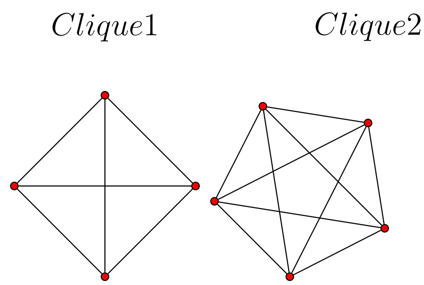
,

$$\mathrm{E}_5(\hat{\theta}_i) = \hat{\theta}_i.$$

Player 5, the central one, is certain about the actions of all other players. Therefore, his $T_5 \in \{1/5, 2/5, 3/5, 4/5, 1\}$ depends on the number of other players taking risky action. For the others, the condition for equilibrium uniqueness is satisfied if $\frac{\alpha}{\beta} < 50\pi - 2$.

This example also outlines how a connection to central players provides additional information to a player. Exchanging private signals with the central player 5 influences expectations about each other player, as it is known that they have observed player 5's signal as well.

A.5 Disconnected cliques



In the case of multiple disconnected cliques, each clique can be treated like a different player, as every member of the clique has the same posterior and will make the same decision. We only need to account for different weights of β. In the case of cliques equal in size, the problem is equivalent to 1st - Morris and Shin example, only accounting for β being weighted l times, where l is the size of each clique. In the example with 2 cliques, for each player in the proper neighborhood:

$$\hat{\theta}_{Clique1} = \frac{\alpha\bar{\theta} + \beta(\sum\limits_{j \in Clique1} x_j)}{\alpha + 4\beta},$$

$$\hat{\theta}_{Clique2} = \frac{\alpha\bar{\theta} + \beta(\sum\limits_{j \in Clique2} x_j)}{\alpha + 5\beta},$$

$$\mathrm{E}_{Clique1}(\hat{\theta}_{Clique2}) = \frac{\alpha\bar{\theta} + 5\beta\hat{\theta}_{Clique1}}{\alpha + 5\beta},$$

$$\mathrm{E}_{Clique2}(\hat{\theta}_{Clique1}) = \frac{\alpha\bar{\theta} + 4\beta\hat{\theta}_{Clique2}}{\alpha + 4\beta}.$$

As players are connected with everyone inside their clique, the condition for the uniqueness of the expectation of their threshold is:

$$\frac{\alpha}{\beta} < -k_i.$$

This is never met with both α and β strictly greater than 0. For a player in Clique 1 regarding players in the other one, this condition is:

$$\frac{\alpha}{\beta} < 2\pi * 5 * 9^2 - k_j.$$

Alternatively, if we treat members of each clique as one representative agent, as proposed above. The condition for each of them becomes

$$\frac{\alpha}{\beta} < 8\pi - 1.$$

As a player in clique i has no information about the private signals in clique j, there will be high uncertainty about the decisions in clique j. Therefore, $\frac{\alpha}{\beta}$ may be higher than in other examples to still have a unique equilibrium. This condition remains valid for any size and number of disconnected cliques that build a network.

Appendix B Generalizations of the model

In this appendix, we follow other scenarios of players' beliefs to show that the proposed model holds even with different specifications of uncertainty players face. We propose the following extensions: non-common priors, private beliefs about β, general payoffs, and heterogeneous edges.

The intuition behind each of those is to provide even stronger differentiation between players. In a very polarized society, an assumption of common priors may be questioned. Alternatively, if some players believe that private signals are inaccurate, they would place close to no weight on them. A situation similar to (Mathevet, 2014) or (Epstein, 2006). The opposite case, where a player updates her posterior only by looking at the private signals, is equally easy to establish by setting her beliefs about private signals to be accurate. In another extension, we allow players to have varying connection strengths to each of their neighbors. An intuitive scenario, where a signal from a close peer is considered more important than one from a distant one.

We also allow for more complex payoff functions and show under what circumstances the results would not change. Our results below prove that the original model may be easily generalized, and the general results remain the same.

B.1 Non-common priors

The first proposed change to the model is the non-common priors. In the context of the current mode, this could be presented as $\bar{\theta}_i$ that is independently drawn for each player from a normal distribution centered around a $\bar{\theta}$ similar to the one from the original model. This could be presented as $\bar{\theta}_i = \bar{\theta} + \zeta_i$, where $\zeta_i \sim \mathcal{N}(0, 1/\gamma)$.

If priors are not shared between players, everyone expects that their own is the proper reflection of the state of the world. The formulas we used so far have been changed only by updating the variances. Instead of α we should use $\frac{\alpha\gamma}{\alpha+\gamma}$. If gamma is getting close to infinity, or the prior becomes the same for every player, the model reverts to its original form. In both cases, formulas stay the same with the substitution of accuracy for $\bar{\theta}$. With

γ being reasonably low, this causes lower weights for the prior and therefore makes reaching equilibrium more probable.

In this model, the condition for the uniqueness of the equilibrium simply changes to

$$\frac{\alpha\gamma}{(\alpha+\gamma)\beta} < 2\pi|\mathcal{N}_j \setminus \mathcal{N}_i|n^2 - k_j.$$

The model's intuition and comparative statics remain the same despite the change in belief formation, as non-common priors only introduce additional uncertainty about the state of the world for players.

B.2 Private beliefs about β

If players do not consider others to obtain the private signal from the same channel as themselves, we may talk of subjective signal functions. The expectations about others' posteriors would have to account for that. It would be expressed in players considering each private signal as having different accuracy. Then, each player would have a vector β_i with their expectation about others' signal strength. Player i's posterior becomes:

$$\hat{\theta}_i = \frac{\alpha\bar{\theta} + \beta_i \circ X\mathcal{G}_i}{\alpha + \sum \beta_i \mathcal{G}_i},$$

where X is the vector of private signals obtained by players and $\mathcal{G}_i$ is a vector from the matrix $\mathcal{G}$ describing i's connections. $\circ$ denotes element-wise multiplication. $\beta_i \cdot \mathcal{G}_i$ is therefore a sum of β_{ij} of player i's neighbours.

In this setting, the condition from Theorem 1 would be less clear but would convey the same message. Let's define

$$\mathcal{G}_{j-i} := \begin{cases} \mathcal{G}_{j,m} - \mathcal{G}_{i,m}, & \text{if } \mathcal{G}_{j,m} - \mathcal{G}_{i,m} > 0, \\ 0, & \text{otherwise,} \end{cases}$$

as a vector of neighbors of j, which are not neighbors of i. The equilibrium of this game is unique if:

$$\frac{\alpha + \beta_i \mathcal{G}_i}{\beta_i \mathcal{G}_{j-i}} < 2\pi n^2.$$

The change from the original theorem is that now, not only is the share of not shared connections important, but also the player's belief in the strength of their signal. It also does not change the comparative statics. The effects of each parameter on the equilibrium follow the same intuition.

B.3 General payoffs

In the model, payoffs of each agent taking a risky action are equal to the share of players deciding to take $a = R$. This assumption is easily generalizable. We can have payoffs in the form of:

$$u_i(a) = \begin{cases} 0, & \text{if } a_i = S, \\ f(k/n) - \theta, & \text{if } a_i = R \text{ and } |\{a_i = R : i \in \mathcal{I}\}| = k. \end{cases}$$

The only difference from the original payoffs is the function f of the share of the players' participation in the risky action. The formula for thresholds T' is then given by:

$$T_i'(T_{-i}') = f\left(\frac{\sum_{j=1}^{n} Pr(\hat{\theta}_j \leq T_j')}{n}\right).$$

For this payoff to work, f has to be increasing. Otherwise, the game does not conserve the main point of having a higher chance of success of the risky action with higher participation. The second condition for the equilibrium uniqueness is the same as in the original model. If $\frac{\partial T_i'}{\partial T_j'}(T_{-i}) < 1$. Any utility function f that satisfies those conditions would have a unique equilibrium and would maintain the idea behind the model. All comparative statics would hold, in terms of the impact of all parameters being positive or negative, for any such f.

B.4 Heterogeneous edges

Players in the game can perceive signals from selected neighbors as more important. In our model, such a mechanism may be represented by heterogeneous edges in the network. To analyze those, we allow $\mathcal{G}$ to assume any values $\mathcal{G}_{i,j} \in \mathbb{R}_+$. We only allow for positive values, as in our case, having a negative $\mathcal{G}_{i,j}$ would mean one player's higher chances of joining the risky action would decrease the likelihood of others. and the game may cease to be of strategic complementarities. Then player i's posterior is given by: $\hat{\theta}_i = \frac{\alpha\bar{\theta}+\beta\mathcal{G}X}{\alpha+\beta\sum\mathcal{G}_i}$, where $\mathcal{G}X$ is the product of matrices $\mathcal{G}$ and X, of all private signals.

The condition for the uniqueness of the equilibrium remains unchanged from the original model. The only difference between the original model and the one here is the size of the effect in Corollary 1. The shift in T_i caused by a shift in private signal from any j in i's neighborhood (including i herself) would be scaled by $\mathcal{G}_{i,j}$. In the original model, players with more extreme signals influence their neighbors more. Increasing the private signal has the same effect as increasing her edge weight and vice versa.

Appendix C Other approaches to endogenous network formation

C.1 Game with choosing nodes

Having a random matching between players willing to establish new edges may occur in some situations (like networking events or apps). In a scenario where players know about each other (e.g., in an organization) but are not aware of each other's opinions, they may be only interested in establishing connections with certain other individuals. Then an edge would only be formed if both parties are interested in establishing it and paying the cost c. The game is similar to a model proposed by (Acemoglu et al., 2014). Its timeline looks similar to the previous section:

0 θ and private signals are drawn by nature

1 Players update their posteriors

2 Players decide whether they want to connect with others

3 Pairs drawn between pairs of players deciding to create edges between each other

4 Edges are created, and private signals exchanged

5 Steps 1-4 are repeated until no more pairs are wanting to establish edges

6 Players decide what decision to take

The theorems from the first section hold here as well. The difference is that players, who in the first section would update their posterior after some iterations of steps 1-4 of the game and may decide not to share their knowledge again, in this context would immediately establish multiple edges. This will lead to denser cliques of players and a more connected graph.

COROLLARY 6. *Player i will want to establish a connection with player j iff:*

$$\frac{e^{-\left(\frac{T_j - \mathrm{E}_i(\hat{\theta}_j)}{(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2}}\right)^2}}{\pi^{1/2}}\Bigg(\big(T_j - \mathrm{E}_i(\hat{\theta}_j)\big)\frac{\alpha + \beta(k_j + |\mathcal{N}_j \setminus \mathcal{N}_i|}{2|\mathcal{N}_j \setminus \mathcal{N}_i|(\alpha + \beta k_j)(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2})}$$
$$-\big(x_i(\alpha + \beta k_j) - (\alpha\bar{\theta} + \beta(\sum_{l \in \mathcal{N}_i \cap \mathcal{N}_j} x_l + |\mathcal{N}_j \setminus \mathcal{N}_i|\hat{\theta}_i))\big)\frac{(2\mathrm{Var}_i(\hat{\theta}_j))^{1/2}}{2|\mathcal{N}_j \setminus \mathcal{N}_i|(\alpha + \beta k_j)}\Bigg) > 0$$

The proof follows the one from the theorem in the main model of the endogenous network, with random connections.

While simulating the evolution on a network with directed connections, the expected result is more edges established, especially at lower levels of θ and x. By construction, players can connect with multiple others in each iteration. Many who would drop out of establishing new edges at some intermediate stage in the random model do connect with all their desired neighbors here. Our results confirm this intuition.

Table 4. Results with c = 0.001 without initial network

$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	93.90	96.66	94.16	47.24	92.0
0.1	95.30	83.96	96.51	91.28	45.88	80.0
0.2	93.55	89.77	93.04	84.42	43.06	74.0
0.3	89.77	91.08	87.83	78.55	40.55	67.0
0.4	70.32	84.48	69.35	54.46	29.58	42.0
0.5	52.83	69.98	54.35	38.73	23.15	24.0
0.6	41.84	60.92	43.08	27.30	18.45	16.0
0.7	20.38	38.26	22.37	10.54	9.95	7.0
0.8	9.60	22.89	8.13	3.00	3.62	3.0
0.9	3.18	5.54	1.82	1.06	0.88	0.0

After introducing the directed connections, more players establish new edges in the first iteration of the network creation game. The more players connected, the stronger the effects observed in the previous section. The intuition does not change, but more players will take the action when it is more costly, losing the ex-post utility.

At low values of θ, we see that players establish so many connections, the model falls out of the unique equilibrium, and the share of risk-takers within the network behaves unpredictably. At higher levels of θ, we see stronger effects of the network than in the random connections model, as players with low private signals are more connected and are more likely to take a risky action.

Table 5. Results with c = 0.001 with initial Erdős-Rényi network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	90.00	83.35	72.88	39.17	88.0
0.1	95.30	86.96	82.63	72.17	38.53	83.0
0.2	93.55	85.57	78.32	66.13	35.60	72.0
0.3	89.77	77.59	79.37	66.64	35.90	58.0
0.4	70.32	76.27	71.53	55.52	31.09	50.0
0.5	52.83	63.16	60.23	43.36	25.61	26.0
0.6	41.84	47.45	43.05	28.27	18.77	21.0
0.7	20.38	31.99	25.84	12.81	11.28	7.0
0.8	9.60	25.91	17.41	8.05	8.20	6.0
0.9	3.18	13.75	6.35	4.56	3.03	0.0

Having a network in place before forming new connections magnifies the effect. It is especially visible with high priors, where the increase in the share of risk-takers is the highest.

Simulations with a random network show a similar pattern to those with no initial network. The number of edges established at low values of prior is again very high, but the intuition remains the same.

Table 6. Results with c = 0.001 with initial Barabási-Albert network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	83.17	37.29	35.65	18.27	84.0
0.1	95.30	80.73	38.69	37.65	18.99	79.0
0.2	93.55	75.24	43.14	40.91	20.67	72.0
0.3	89.77	71.25	44.82	41.95	21.07	56.0
0.4	70.32	74.84	43.05	39.92	20.19	55.0
0.5	52.83	54.49	48.59	40.99	21.67	35.0
0.6	41.84	49.33	47.14	39.68	21.33	19.0
0.7	20.38	29.51	45.66	41.09	21.67	8.0
0.8	9.60	31.02	44.29	42.48	21.90	10.0
0.9	3.18	13.84	44.29	42.88	22.12	4.0

Similarly to the random connections, having some players with very high centrality increases the likelihood of the formation of cliques of players with posteriors varying significantly from the actual realization of the fundamentals.

Players establishing connections try to convince others to join the conflict. Their decision depends on their targets' position in the network. They have very specific targets for their new edges, and because of that, more mismatches happen. It decreases the number of established connections compared to the previous networks.

At low levels of prior $\bar{\theta}$, a player will be less likely to connect. The risky action is expected to deliver high utility. Additional information does not change players' actions and provides little value to them. Network effects are the strongest with fewer players deciding on risky action at low $\bar{\theta}$ and more at high. As players gain higher ex-ante utility from a connection with others who have fewer shared connections with them, the game maintains low network connectedness and meets the condition for a unique equilibrium.

C.2 Information rooms

To better describe an online phenomenon of groups or forums, where people usually hold similar beliefs, we introduce a version of the game where players are faced with a decision to join cliques. Within a connected group, players exchange information with each other. We have m “rooms”. A player may belong to any of those. Initially, there may or may not be any players in any of the groups, and players will choose whether to join a group or not. If there is an initial network and some players are members of information rooms, others choose between all of them. The timeline of this game is shorter than before, as there is only one iteration of joining the rooms:

0 θ and private signals are drawn by nature
1 Players update their posteriors
2 Players decide to join any of m groups
3 Players are assigned to groups
4 Edges created between players within groups and private signals exchanged
5 Players decide what decision to take

If we have initial rooms established and players decide to join a clique, the decision is following Theorem 2. The total expected gain from joining the group is outlined in the proof of Theorem 2, summed over all its members. If no initial group membership is present, players take the average of the population and decide whether they want to exchange information. Similarly to the random connections section. Therefore, the reasoning behind the decision to join the information exchange from the first section holds. Only the aggregation of expected utility differs.

Creating information rooms gives players the possibility to connect to a very large number of others at once and at a very low cost. We can expect that simulating such a game would lead to very high connectedness of the networks and situations where the condition for the uniqueness of the equilibrium is not met.

Table 7. Results with c = 0.001 without initial network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	93.43	74.62	74.62	37.55	15.0
0.1	95.30	84.01	90.29	90.82	45.38	52.0
0.2	93.55	89.94	89.31	89.89	44.76	66.0
0.3	89.77	87.11	85.97	86.23	43.06	67.0
0.4	70.32	73.20	96.08	96.26	48.07	89.0
0.5	52.83	62.26	95.72	96.47	48.12	75.0
0.6	41.84	58.36	98.36	98.73	49.25	89.0
0.7	20.38	35.93	93.72	93.78	46.87	84.0
0.8	9.60	21.05	95.98	95.91	47.98	96.0
0.9	3.18	7.97	71.70	71.72	35.97	72.0

The introduction of information rooms makes the effect of an endogenous network stronger than in the case of choosing connections. Players immediately get more connections at a lower cost, and we see more connected graphs, especially at higher $\bar{\theta}$.

Table 8. Results with c = 0.001 with initial Erdős-Rényi network						
$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	90.29	95.25	95.49	47.75	86.0
0.1	95.30	86.98	98.64	98.75	49.36	95.0
0.2	93.55	86.23	93.63	93.68	46.79	69.0
0.3	89.77	76.92	96.07	96.11	48.06	76.0
0.4	70.32	76.00	98.97	98.97	49.49	99.0
0.5	52.83	64.05	98.52	98.38	49.23	91.0
0.6	41.84	45.32	96.87	97.28	48.47	91.0
0.7	20.38	34.00	97.50	97.76	48.91	87.0
0.8	9.60	27.00	99.02	99.01	49.51	99.0
0.9	3.18	16.03	93.18	93.16	46.59	93.0

The Erdős-Rényi network makes the effect of information rooms the strongest. We can see the highest share of connected graphs and high levels of beta index across all fundamentals. This, in turn, causes fewer players to take risk-free actions and more to take risky ones.

Table 9. Results with c = 0.001 with initial Barabási-Albert network

$\bar{\theta}$	initial risk-takers %	risk-takers % with network	avg. degree with x $\leq \bar{\theta}$	avg. degree with x $> \bar{\theta}$	β index	% connected graphs
0.0	99.00	83.00	99.04	99.02	49.52	99.0
0.1	95.30	80.99	97.86	97.48	48.82	91.0
0.2	93.55	75.00	98.92	98.81	49.44	99.0
0.3	89.77	70.98	99.03	99.02	49.52	99.0
0.4	70.32	76.00	98.88	98.79	49.43	99.0
0.5	52.83	54.00	98.77	98.86	49.38	97.0
0.6	41.84	50.17	95.45	95.97	47.64	93.0
0.7	20.38	29.12	97.07	97.06	48.55	97.0
0.8	9.60	31.06	95.85	95.57	47.90	92.0
0.9	3.18	14.48	75.72	74.70	37.70	73.0

The Barabási-Albert network case behaves similarly to the Erdős-Rényi case. However, the effects are smaller. We can still see a significant increase in risk-takers after the network introduction at $\bar{\theta} = 0.9$ and a decrease at $\bar{\theta} = 0$. The connectedness of the network is lower than in the case of the Erdős-Rényi.

The number of established connections is greater than in other network formation models. The network is highly connected. Surprisingly, at low levels of $\bar{\theta}$ with no initial network, fewer players join multiple information rooms, as more of them are convinced of the success of risky action and decide not to bear more interaction costs.

Appendix D Simulation code in Python 3

```
import matplotlib.pyplot as plt
import networkx as nx
import numpy as np
import random as rn
import scipy.stats as sp


class GlobalGame:
    def __init__(self, n, theta_bar, alpha, beta):
        """
        This code simulates a global game on a network.

        :param n: Number of players
        :param theta_bar: Mean of the distribution of fundamentals
        :param alpha: Precision of theta_bar
        :param beta: Precision of private signal
        """
        self.n = n
        self.theta_bar = theta_bar
        self.alpha = alpha
        self.beta = beta

        self.theta = rn.gauss(self.theta_bar, np.sqrt(1.0 / self.alpha))

        self.G = np.eye(n, n)

        self.x = np.empty(n)
        self.count_x()
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        self.theta_hat = np.empty(n)
        self.count_theta_hat()

        self.e_theta_hat = np.diag(self.theta_hat)
        self.e_theta_hat.setflags(write=True)
        self.var_theta_hat = np.zeros((n, n))
        self.count_expectations()

        self.T = np.random.rand(n)
        self.count_t()

        self.a = np.zeros(n, dtype=bool)
        self.count_a()

        self.expected_utility = np.zeros(n)
        self.count_utility()

        self.groups = {}

    def count_x(self):
        stdev = np.sqrt(1.0 / self.beta)
        for i in range(self.n):
            self.x[i] = rn.gauss(self.theta, stdev)

    def count_theta_hat(self):
        for i in range(self.n):
            self.theta_hat[i] =
                (self.alpha * self.theta_bar + self.beta *
                np.dot(self.G[i, :], self.x)) / \
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            (self.alpha + self.beta * sum(self.G[i, :]))

    def count_expectations(self):
        for i in range(self.n):
            for j in range(self.n):
                if i == j:
                    continue
                common = np.logical_and(self.G[i, :], self.G[j, :])
                k_j = sum(self.G[j, :])
                self.e_theta_hat[i, j] =
                    (self.alpha * self.theta_bar + self.beta *
                    (np.dot(common, self.x) + (k_j - sum(common)) *
                    self.theta_hat[i])) /
                    (self.alpha + self.beta * k_j)
                self.var_theta_hat[i, j] =
                    self.beta * (k_j - sum(common)) / \
                    (self.alpha + self.beta * k_j)

    def count_t(self, iterations=1000):
        i = 0
        while i < iterations:
            oldt = self.T
            for i in range(self.n):
                mask = np.ones(self.n, dtype=bool)
                mask[i] = 0
                self.T[i] =
                    (sum(map(lambda x: sp.norm.cdf(x),
                    np.divide(self.T[mask] -
                    self.e_theta_hat[i, mask],
                    self.var_theta_hat[i, mask]))) + 1) / self.n
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            i += 1
            if sum([abs(oldt[x] - self.T[x])
                for x in range(self.n)]) / self.n < 0.001:
                break

    def count_a(self):
        for i in range(self.n):
            if self.theta_hat[i] < self.T[i]:
                self.a[i] = 1
            else:
                self.a[i] = 0

    def count_utility(self):
        for i in range(self.n):
            self.expected_utility[i] = self.T[i] - self.theta_hat[i]

    def generate_ba_network(self):
        self.G[0, 1] = 1
        self.G[1, 0] = 1
        for i in range(2, self.n):
            denominator = sum(sum(self.G[:i, :i]))
            for j in range(i):
                if rn.random() < sum(self.G[j, :]) / denominator:
                    self.G[i, j] = 1
                    self.G[j, i] = 1

    def generate_er_network(self, p):
        for i in range(self.n):
            for j in range(i, self.n):
                if rn.random() < p:
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            self.G[i, j] = 1
            self.G[j, i] = 1

def endogenous_network_random(self, c, turns):
    for t in range(turns):
        to_add = []
        conn = 0
        for i in range(self.n):
            mask = np.logical_not(self.G[i, :])
            k = sum(self.G[:, mask])
            common =
                np.logical_and(self.G[mask, :],
                    self.G[i, :])
            nn = k - np.sum(common)
            expected_benefit =
                sum(np.exp(-((self.T[mask] -
                self.e_theta_hat[i, mask]) /
                np.sqrt(2 *
                self.var_theta_hat[i, mask])) ** 2) /
                np.sqrt(np.pi) * ((self.T[mask] -
                self.e_theta_hat[i, mask]) *
                (self.alpha + self.beta * (k + nn))) /
                np.sqrt(2 * self.var_theta_hat[i, mask]) -
                (self.x[i] * (self.alpha + self.beta * k) -
                (self.alpha * self.theta_bar + self.beta *
                (np.dot(common, self.x) +
                (nn * self.theta_hat[i])))) *
                np.sqrt(2 * self.var_theta_hat[i, mask]) /
                2 * nn * (self.alpha + self.beta * k))) /
                sum(mask)
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            if expected_benefit > c and
                self.expected_utility[i] +
                expected_benefit > c and self.T[i] < 1:
                to_add.append(i)
        if len(to_add) < 2:
            break
        while len(to_add) > 1:
            i = rn.choice(to_add)
            to_add.remove(i)
            non_neighbours =
                [x for x in to_add if self.G[i, x] == 0]
            if len(non_neighbours) < 1:
                continue
            j = rn.choice(non_neighbours)
            to_add.remove(j)
            self.G[i, j] = 1
            self.G[j, i] = 1
            conn += 1
        if conn == 0:
            self.run_information_exchange()
            break
        self.run_information_exchange()


def endogenous_network_directed(self, c):
    new = 1
    while new > 0:
        to_add = {}
        for i in range(self.n):
            to_add[i] = []
            mask = np.logical_not(self.G[i, :])
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        k = sum(self.G[:, mask])
        common = np.logical_and(self.G[mask, :],
             self.G[i, :])
        nn = k - np.sum(common)
        expected_benefit =
             np.array(np.exp(-((self.T[mask] -
             self.e_theta_hat[i, mask]) / np.sqrt(2 *
             self.var_theta_hat[i, mask])) ** 2) /
             np.sqrt(np.pi) * ((self.T[mask] -
             self.e_theta_hat[i, mask]) *
             (self.alpha + self.beta * (k + nn))) /
             np.sqrt(2 * self.var_theta_hat[i, mask]) -
             (self.x[i] * (self.alpha + self.beta * k) -
             (self.alpha * self.theta_bar + self.beta *
             (np.dot(common, self.x) +
             (nn * self.theta_hat[i]))))) * \
             np.sqrt(2 * self.var_theta_hat[i, mask]) /
             (2 * nn * (self.alpha + self.beta * k)))
        ind = 0
        for j in range(self.n):
             if self.G[i, j] == 0:
                  if expected_benefit[ind] > c and
                  self.expected_utility[i] +
                  expected_benefit[ind] > c and \
                            self.T[i] < 1:
                       to_add[i].append(j)
                  ind += 1
    new = 0
    for k, v in to_add.items():
        for j in v:
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                    if k in to_add[j]:
                        new += 1
                        self.G[k, j] = 1
                        self.G[j, k] = 1
                        to_add[j].remove(k)
                to_add[k] = []
            self.run_information_exchange()

    def endogenous_network_groups(self, c, new_groups=0):
        for ng in range(len(self.groups),
        len(self.groups) + new_groups):
            self.groups[ng] = []
        to_join = {x: [] for x in self.groups.keys()}
        for i in range(self.n):
            if all([len(x) == 0 for x in self.groups.values()]):
                mask = np.logical_not(self.G[i, :])
                k = sum(self.G[:, mask])
                common = np.logical_and(
                    self.G[mask, :], self.G[i, :])
                nn = k - np.sum(common)
                expected_benefit = sum(np.exp(-((self.T[mask] -
                    self.e_theta_hat[i, mask]) / np.sqrt(
                    2 * self.var_theta_hat[i, mask])) ** 2) /
                    np.sqrt(np.pi) * ((self.T[mask] -
                    self.e_theta_hat[i, mask]) *
                    (self.alpha + self.beta * (k + nn))) /
                    np.sqrt(2 * self.var_theta_hat[i, mask]) -
                    (self.x[i] * (self.alpha + self.beta * k) -
                    (self.alpha * self.theta_bar + self.beta *
                    (np.dot(common, self.x) + (nn *
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                self.theta_hat[i])))) * np.sqrt(2 *
                self.var_theta_hat[i, mask]) / (
                2 * nn * (self.alpha + self.beta * k))) /
                sum(mask)
            if expected_benefit > c and
            self.expected_utility[i] + expected_benefit > c
            and self.T[i] < 1:
                to_join[rn.choice(list(self.groups.keys()))].
                    append(i)
        else:
            for g in self.groups.keys():
                mask = [self.G[i, j] == 0 and j in
                    self.groups[g] for j in range(self.n)]
                k = sum(self.G[:, mask])
                common = np.logical_and(self.G[mask, :],
                    self.G[i, :])
                nn = k - np.sum(common)
                expected_benefit =
                    sum(np.exp(-((self.T[mask] -
                    self.e_theta_hat[i, mask]) /
                    np.sqrt(2 *
                    self.var_theta_hat[i, mask])) ** 2) /
                    np.sqrt(np.pi) * ((self.T[mask] -
                    self.e_theta_hat[i, mask]) *
                    (self.alpha + self.beta * (k + nn))) /
                    np.sqrt(2 *
                    self.var_theta_hat[i, mask]) -
                    (self.x[i] * (self.alpha + self.beta * k) -
                    (self.alpha * self.theta_bar + self.beta *
                    (np.dot(common, self.x) +
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                    (nn * self.theta_hat[i])))) *
                    np.sqrt(2 * self.var_theta_hat[i, mask])
                    / (2 * nn *
                    (self.alpha + self.beta * k))) /
                    sum(mask)
                if expected_benefit > c and
                self.expected_utility[i] + expected_benefit >
                c and self.T[i] < 1:
                    to_join[g].append(i)
    for k, v in to_join.items():
        self.groups[k] += v
    for v in self.groups.values():
        for i in range(len(v) - 1):
            for j in range(i + 1, len(v)):
                self.G[v[i], v[j]] = 1
                self.G[v[j], v[i]] = 1
    self.run_information_exchange()

def draw_ie_groups(self, n_groups, prob_membership):
    self.groups = {x: [] for x in range(n_groups)}
    for i in range(self.n):
        for x in range(n_groups):
            if rn.random < prob_membership:
                self.groups[x].append(i)
    for v in self.groups.values():
        for i in range(len(v) - 1):
            for j in range(i + 1, len(v)):
                self.G[v[i], v[j]] = 1
                self.G[v[j], v[i]] = 1
    self.run_information_exchange()
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    def run_information_exchange(self):
        self.count_theta_hat()
        self.e_theta_hat = np.diag(self.theta_hat)
        self.count_expectations()
        self.count_t()
        self.count_a()
        self.count_utility()

    def reset_network(self):
        self.G = np.eye(self.n, self.n)

    def draw_graph(self):
        g = nx.from_numpy_matrix(self.G)
        nx.draw_networkx(g, node_color=self.a)

    def save_graph(self, filename):
        g = nx.from_numpy_matrix(self.G)
        nx.draw_networkx(g, node_color=self.a)
        plt.tight_layout()
        plt.savefig(filename, format="PNG")
        plt.close()

    def draw_new_theta(self):
        self.theta = rn.gauss(self.theta_bar,
            np.sqrt(1.0 / self.alpha))
        self.count_x()

    def set_theta(self, new_theta):
        self.theta = new_theta
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        self.count_x()

    def make_edge(self, i, j):
        self.G[i, j] = 1
        self.G[j, i] = 1

    def get_degrees(self):
        deg = []
        for i in range(self.n):
            deg.append(sum(self.G[i, :]))
        return deg

    def beta_index(self):
        return float(sum(sum(self.G))) / (2 * self.n)

    def is_connected(self):
        return nx.is_connected(nx.Graph(self.G))

    def k_core_size(self):
        return nx.k_core(nx.Graph(self.G - np.eye(self.n, self.n))).
        number_of_nodes()

    def rich_club_coef(self):
        return nx.rich_club_coefficient(nx.Graph(self.G -
        np.eye(self.n, self.n)))
```